Existence of 1-harmonic map flow

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joint work with L. Giacomelli and S. Moll

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$$X(\Omega, \mathcal{N}) = \{\boldsymbol{u} \in X(\Omega, \mathbb{R}^N) \text{ s.t. } \boldsymbol{u}(\boldsymbol{x}) \in \mathcal{N} \text{ for a.e. } \boldsymbol{x} \in \mathcal{N}\}$$

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1-harmonic map flow — L^2-gradient flow of constrained total variation functional, given for $\boldsymbol{u} \in C^1(\Omega, \mathcal{N})$ by

$$TV^{\mathcal{N}}_{\Omega}(\boldsymbol{u}) = \int_{\Omega} |\nabla \boldsymbol{u}|$$

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for $\boldsymbol{u} \in C^1(\mathcal{M}, \mathcal{N})$ the total variation is now given by

$$TV^{\mathcal{N}}_{\mathcal{M}}(\boldsymbol{u}) = \int_{\mathcal{M}} |\nabla \boldsymbol{u}|_\gamma = \int_{\mathcal{M}} \left(\gamma^{\alpha\beta} \boldsymbol{u}_{x^\alpha} \boldsymbol{u}_{x^\beta} \right)^{\frac{1}{2}}$$

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for $p > 1$, p-harmonic flow — L^2-gradient flow of $\frac{1}{p}\int_{\mathcal{M}} |\nabla \boldsymbol{u}|_{\gamma}^{p}$

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$$\boldsymbol{u}_t = \mathrm{div}(|\nabla \boldsymbol{u}|^{p-2}\nabla \boldsymbol{u}) + \mathcal{A}_{\mathcal{N}}(\boldsymbol{u})(\boldsymbol{u}_{x^i}, \boldsymbol{u}_{x^i}) \qquad \text{(pHMFE)}$$

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energy inequality

$$\frac{1}{p}\int_{\mathcal{M}} |\nabla \boldsymbol{u}|_\gamma^p + \int_0^t \int_{\mathcal{M}} |\boldsymbol{u}_t|_\gamma^2 \leq \frac{1}{p}\int_{\mathcal{M}} |\nabla \boldsymbol{u}_0|_\gamma^p \qquad \text{(EI)}$$

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$\mathcal{M}$, $\mathcal{N}$ – compact, $\mathcal{R}_{\mathcal{N}}$ — Riemann curvature tensor of $\mathcal{N}$

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Theorem (Eells-Sampson, 1964)

Let $p = 2$ and $\boldsymbol{u}_0 \in C^\infty(\mathcal{M}, \mathcal{N})$. If $\mathcal{R}_{\mathcal{N}} \leq 0$, unique smooth harmonic map flow $\boldsymbol{u}$ starting with $\boldsymbol{u}_0$ exists for all $t > 0$. There exists a sequence (t_i) such that $(\boldsymbol{u}(t_i))$ converges uniformly to a harmonic map $\boldsymbol{u}_*$.

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Theorem (Chen-Struwe, 1988)

Let $p = 2$. For any $\boldsymbol{u}_0 \in C^\infty(\mathcal{M}, \mathcal{N})$ there exists a global weak solution to (pHMFE) *with initial datum $\boldsymbol{u}_0$ satisfying* (EI). *There exists a sequence (t_i) such that $(\boldsymbol{u}(t_i))$ converges weakly in $W^{1,2}(\mathcal{M}, \mathcal{N})$ to a weakly harmonic map $\boldsymbol{u}_*$.*

$p \neq 2$, part I: Hungerbühler

Theorem (Hungerbühler, 1997)

Let $p = m$*,* $\boldsymbol{u}_0 \in W^{1,p}(\mathcal{M}, \mathcal{N})$*. There exists a global weak solution to* (pHMFE) *with initial datum* $\boldsymbol{u}_0$ *satisfying* (EI)*. This solution is regular except finitely many time instances. There is at most one solution satisfying* $\nabla \boldsymbol{u} \in L^\infty(]0, \infty[\times \mathcal{M})$*.*

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Theorem (Hungerbühler, 1996)

Let $\mathcal{N}$ *be a homogeneous space and let* $\boldsymbol{u}_0 \in W^{1,p}(\mathcal{M}, \mathcal{N})$*. There exists a global weak solution to* (pHMFE) *with initial datum* $\boldsymbol{u}_0$ *satisfying* (EI)*.*

$p \neq 2$, part 2: Fardoun & Regbaoui

Theorem (Fardoun-Regbaoui, 2002-2003)

Let $\boldsymbol{u}_0 \in C^{2+\alpha}(\mathcal{M}, \mathcal{N})$. If $\mathcal{R}_{\mathcal{N}} \leq 0$ or $\int_{\mathcal{M}} |\nabla \boldsymbol{u}|_\gamma^p$ is small enough, there exists a regular global weak solution to (pHMFE) *with initial datum $\boldsymbol{u}_0$. There exists a sequence (t_i) such that $(\boldsymbol{u}(t_i))$ converges uniformly to a p-harmonic map $\boldsymbol{u}_*$.*

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- $SO(3)$ or $SE(3)$ — orientations of objects (e. g. camera trajectories)
- $SPD(3)$ — diffusion tensor space

The Euclidean case $\mathcal{N} = \mathbb{R}^n = \mathbb{R}^N$

$$TV_\Omega(\boldsymbol{u}) = \int_\Omega |\nabla \boldsymbol{u}| = \sup \left\{ \int_\Omega \boldsymbol{u} \cdot \mathrm{div} \boldsymbol{\varphi} \colon \boldsymbol{\varphi} \in C_c^1(\Omega), |\boldsymbol{\varphi}| \leq 1 \right\}$$

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$\boldsymbol{u}$ satisfies energy inequality

$$\int_\Omega |\nabla \boldsymbol{u}(t,\cdot)| + \int_0^t \int_\Omega \boldsymbol{u}_t^2 \leq \int_\Omega |\nabla \boldsymbol{u}_0|$$

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Theorem (Andreu-Ballester-Caselles-Mazon, 2000)

$\boldsymbol{u} \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;BV(\Omega))$ *is a steepest descent curve of* TV_Ω *iff there exists* $\boldsymbol{Z} \in L^\infty(]0,T[\times\Omega)$ *with* $\operatorname{div}\boldsymbol{Z} \in L^2(]0,T[\times\Omega)$ *such that*

$$\boldsymbol{u}_t = \operatorname{div}\boldsymbol{Z} \quad \textit{a. e. in } \Omega,$$

$$(\nabla\boldsymbol{u},\boldsymbol{Z}) = |\nabla\boldsymbol{u}| \quad \textit{as measures on } \Omega,$$

$$|\boldsymbol{Z}| \leq 1 \quad \textit{a. e. in } \Omega$$

$$\boldsymbol{Z}\cdot\boldsymbol{\nu}^\Omega = 0 \quad \textit{a. e. on } \partial\Omega$$

for a. e. $t \in]0,T[$

Regular 1-harmonic map flow

formally

$$\boldsymbol{u}_t = \pi_{\mathcal{N}}(\boldsymbol{u})\operatorname{div}\frac{\nabla \boldsymbol{u}}{|\nabla \boldsymbol{u}|} \quad \left(+ \quad \frac{\nabla \boldsymbol{u}}{|\nabla \boldsymbol{u}|} \cdot \boldsymbol{\nu}^{\Omega} = 0\right) \tag{1HMFE}$$

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Definition

We say that $\boldsymbol{u} \in W^{1,2}(]0,T[\times\Omega,\mathcal{N})$ with $\nabla\boldsymbol{u} \in L^{\infty}(]0,T[\times\Omega)$ is a regular solution to (1HMFE) if there exists $\boldsymbol{Z} \in L^{\infty}(]0,T[\times\Omega)$ such that

$$\boldsymbol{u}_t = \mathrm{div}\boldsymbol{Z},$$

$$\boldsymbol{Z} \in T_{\boldsymbol{u}}\mathcal{N}, \quad |\boldsymbol{Z}| \leq 1, \quad \boldsymbol{Z} = \frac{\nabla \boldsymbol{u}}{|\nabla \boldsymbol{u}|} \text{ if } \nabla\boldsymbol{u} \neq 0 \quad \text{a. e. in }]0,T[\times\Omega,$$

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Existence of regular 1-harmonic map flow

Theorem (Giga-Kashima-Yamazaki, 2004)

Suppose that $\mathcal{N}$ *is compact,* $\Omega = \mathbb{T}^m$*,* $\boldsymbol{u}_0 \in C^{2+\alpha}(\Omega, \mathcal{N})$ *and* $\|\nabla \boldsymbol{u}_0\|_{L^p(\Omega)}$ *is small enough for some* $p > 1$*. There exists a local-in-time regular solution to* (1HMFE) *with initial datum* $\boldsymbol{u}_0$ *satisfying* (EI)*.*

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Theorem (Giacomelli-Ł-Moll, preprint 2017)

Suppose that $\mathcal{N}$ *is a closed submanifold in* $\mathbb{R}^N$ *and* Ω *is a convex domain in* $\mathbb{R}^m$*. If* $\boldsymbol{u}_0 \in W^{1,\infty}(\Omega, \mathcal{N})$*, there exists a unique local-in-time regular solution to* (1HMFE) *with initial datum* $\boldsymbol{u}_0$ *satisfying* (EI)*.*

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If $\mathcal{R}_{\mathcal{N}} \leq 0$ or the image of the datum is small enough, the solution is global and becomes constant in finite time.

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  - h restricted to $\mathcal{N}$ coincides with g
  - h coincides with the Euclidean metric outside a neighbourhood of $\mathcal{N}$
  - there exists a neighbourhood U of $\mathcal{N}$ in $\mathbb{R}^N$ and an involution i of U that is isometric with respect to h

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- Bochner's formula

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla \boldsymbol{u}|^2 = \mathrm{div}(\boldsymbol{u}_{x^i} \cdot \boldsymbol{Z}_{x^i}) - (\pi_{\mathcal{N}}(\boldsymbol{u})\boldsymbol{u}_{x^i x^j}) \cdot \boldsymbol{Z}_{i,x^j} + \boldsymbol{Z}_i \cdot \mathcal{R}_{\mathcal{N}}(\boldsymbol{u})(\boldsymbol{u}_{x^i}, \boldsymbol{u}_{x^j})\boldsymbol{u}_{x^j}$$

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- due to convexity of Ω, from Bochner's formula we get uniform estimate

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla \boldsymbol{u}|^p \leq C(\mathcal{N})\int_\Omega |\nabla \boldsymbol{u}|^{p+1},$$

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- standard limit passage

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for the gradient flow of $\int |u_x| + |u_y|$ there is a non-convex polygon Ω and $u_0 \in W^{1,\infty}(\Omega)$ such that $u(t,\cdot) \notin W^{1,1}_{loc}(\Omega)$ for small $t > 0$ (Ł-Moll-Mucha, 2017)

Manifold domain

Theorem (Giacomelli-Ł-Moll, preprint 2017)

Suppose that $\mathcal{N}$ *is a closed submanifold in* $\mathbb{R}^N$ *and* $\mathcal{M}$ *is a compact, orientable Riemannian manifold. If* $\boldsymbol{u}_0 \in W^{1,\infty}(\mathcal{M}, \mathcal{N})$*, there exists a unique local-in-time regular solution to* (1HMFE) *with initial datum* $\boldsymbol{u}_0$ *satisfying* (EI)*.*

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If $\mathcal{R}_N \leq 0$*, the solution is global. If furthermore* $\mathcal{R}ic_{\mathcal{M}} \geq 0$*, the solution converges uniformly to a* 1*-harmonic map.*

BV solutions ($\mathcal{N}$ - hyperoctant)

Theorem (Giacomelli-Mazon-Moll, 2013-2014)

Let $\mathcal{N}$ be a hyperoctant of $\mathbb{S}^n$ and $\boldsymbol{u}_0 \in BV(\Omega, \mathcal{N})$. There exists a solution to

$$\boldsymbol{u}_t = \mathrm{div}\boldsymbol{Z} + \boldsymbol{u}^g |\nabla \boldsymbol{u}| \quad \textit{as measures on } \Omega,$$

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in a. e. $t \in]0, T[$ for arbitrarily large $T > 0$.

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there holds $(\nabla \boldsymbol{u}, \boldsymbol{Z}) = |\boldsymbol{u}^*||\nabla \boldsymbol{u}|$ as measures for a. e. $t \in]0, T[$.

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Theorem (Giacomelli-Ł, preprint 2018)

Let $\boldsymbol{u} \in H^1(0, \infty; L^2(I)^n) \cap L^\infty(0, \infty; BV(I)^n)$ *be the steepest descent curve of* TV_I *emanating from* $\boldsymbol{u}_0 \in BV(I)^n$*. There holds*

$$|\boldsymbol{u}_x(t, \cdot)| \le |\boldsymbol{u}_{0,x}|$$

as measures for $t > 0$*.*

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- pass to the limit $\varepsilon \to 0^+$, then $p \to 1^+$, $R \to r^+$, relax initial datum

Completely local estimates

Theorem (Bonforte-Figalli, 2012)

Let u *be a solution to the scalar total variaton flow with initial datum* $u_0 \in BV(I)$. *Then* $|u_x|(\{x_0\}) \leq |u_{0,x}|(\{x_0\})$ *for any* $x_0 \in J_{u_0}$ *and* $\mathrm{osc}_A u \leq \mathrm{osc}_A u_0$ *on any interval* $A \subset I$ *where* u_0 *is continuous.*

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Theorem (Briani-Chambolle-Novaga-Orlandi, 2012)

Let Ω be an open domain in $\mathbb{R}^n$ and let $\boldsymbol{u}_0 \in L^2(\Omega, \mathbb{R}^n)$ be such that $\operatorname{div} \boldsymbol{u}_0$ is a Radon measure on Ω. The L^2-gradient flow of functional $\int_\Omega |\operatorname{div} \boldsymbol{u}|$ satisfies $(\operatorname{div} \boldsymbol{u}(t,\cdot))_\pm \leq (\operatorname{div} \boldsymbol{u}_0)_\pm$.

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for a solution u to the scalar TV flow with initial datum $u_0 \in BV(\Omega)$, does $|\nabla^s u(t,\cdot)| \leq |\nabla^s u_0|$?

A note about TV flow for $m = 1$

take $\boldsymbol{u} \in BV(I)^n$, $\boldsymbol{Z} \in W^{1,1}(I)^n$ with $|\boldsymbol{Z}| \leq 1$ a. e.

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the condition $(\boldsymbol{u}_x, \boldsymbol{Z}) = |\boldsymbol{u}_x|$ is equivalent to

$$\boldsymbol{Z} = \frac{\boldsymbol{u}_x}{|\boldsymbol{u}_x|} \quad |\boldsymbol{u}_x| - \text{a. e. in } I$$

(a measure derivative)

1-harmonic map flow for $m = 1$

Definition

Suppose that $\boldsymbol{u} \in W^{1,2}(0,T;L^2(I,\mathcal{N})) \cap L^\infty(0,T;BV(I,\mathcal{N}))$ and $\operatorname{dist}_g(\boldsymbol{u}_-,\boldsymbol{u}_+) < \operatorname{inj}\mathcal{N}$ on $J_{\boldsymbol{u}}$.

We say that $\boldsymbol{u}$ is a solution to (1HMFE) if there exists $\boldsymbol{Z} \in L^\infty(]0,T[\times I)^n$ such that a. e. in $]0,T[$ there holds

$$\boldsymbol{u}_t = \pi_{\mathcal{N}}(\boldsymbol{u})\boldsymbol{Z}_x \quad \text{a. e. on } I,$$

$$\boldsymbol{Z} \in T_{\boldsymbol{u}}\mathcal{N}, \quad |\boldsymbol{Z}| \le 1 \quad \text{a. e. on } I,$$

$$\boldsymbol{Z} = \frac{\boldsymbol{u}_x}{|\boldsymbol{u}_x|} \quad |\boldsymbol{u}_x| - \text{a. e. on } I \setminus J_{\boldsymbol{u}},$$

$$\boldsymbol{Z}^- = T(\boldsymbol{u}^-), \boldsymbol{Z}^+ = T(\boldsymbol{u}^+) \quad \text{on } J_{\boldsymbol{u}},$$

$$\boldsymbol{Z} \cdot \boldsymbol{\nu}^\Omega = 0 \quad \text{on } \partial I$$

1-harmonic flow for $m = 1$

Theorem (Giacomelli-Ł, in preparation)

Let $\boldsymbol{u}_0 \in BV(I, \mathcal{N})$ *satisfy* $\mathrm{dist}_g(\boldsymbol{u}_0^-, \boldsymbol{u}_0^+) < R_*$ *on* $J_{\boldsymbol{u}}$*,* $R_* = R_*(\mathcal{N})$*.*
For any $T > 0$ *there exists a solution to* (1HMFE) *starting with* $\boldsymbol{u}_0$*.*

Relaxed TV

for $\boldsymbol{u} \in BV(I, \mathcal{N})$, define

$$TV_g(\boldsymbol{u}) =$$
$$\inf \left\{ \liminf \int_I |\boldsymbol{u}_x^k| \colon (\boldsymbol{u}^k) \subset W^{1,\infty}(I, \mathcal{N}), \boldsymbol{u}^k \overset{*}{\rightharpoonup} \boldsymbol{u} \text{ in } BV(I, \mathcal{N}) \right\}$$

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there holds

$$TV_g(\boldsymbol{u}) = \int_I |\boldsymbol{u}_x|_g,$$

where

$$|\boldsymbol{u}_x|_g = |\boldsymbol{u}_x| \llcorner I \setminus J_{\boldsymbol{u}} + \operatorname{dist}_g(\boldsymbol{u}_-, \boldsymbol{u}_+)\mathcal{H}^0 \llcorner J_{\boldsymbol{u}}$$

(Giaquinta-Mucci, 2006)

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  the nonlinear term depends on Z (no sphere trick)
- lack of strong convergence of Z — cannot pass to the limit timeslice-wise

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- calculate $\frac{\boldsymbol{u}_x}{|\boldsymbol{u}_x|}$ by chain rule

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- take $\varphi \geq 0$ — cutoff centered around the jump:

$$\int_I |\boldsymbol{u}_x|\varphi = \int_I \widetilde{g}_{ij}(\boldsymbol{u})\, \widetilde{z}^i\, \widetilde{u}^j_x \varphi = -\int_I \widetilde{g}_{ij,k}(\boldsymbol{u})\, \widetilde{u}^k_x\, \widetilde{z}^i\, \widetilde{u}^j \varphi$$

$$-\int_I \widetilde{g}_{ij}(\boldsymbol{u})\, \widetilde{u}^i_t\, \widetilde{u}^j \varphi + \int_I \widetilde{g}_{ij}(\boldsymbol{u})\, \widetilde{\Gamma}^i_{jk}(\boldsymbol{u})\, \widetilde{z}^j\, \widetilde{u}^k_x\, \widetilde{u}^j \varphi - \int_I \widetilde{g}_{ij}(\boldsymbol{u})\, \widetilde{z}^i\, \widetilde{u}^j \varphi_x$$

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- slice-wise estimate

$$\mathrm{dist}(\boldsymbol{u}(t,x), \gamma_{\boldsymbol{u}(t,a),\boldsymbol{u}(t,b)}) \leq C \int_a^b |\boldsymbol{u}_t(t,\cdot)|$$

for $x \in]a,b[$, $t > 0$, where $\gamma_{\boldsymbol{u}(t,a),\boldsymbol{u}(t,b)}$ is the minimal geodesic joining $\boldsymbol{u}(t,a)$ and $\boldsymbol{u}(t,b)$

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- maximum principle in a convex ball
- relaxation estimate

$$\liminf \int_I |\boldsymbol{u}_x^k|\varphi \geq \int_I |\boldsymbol{u}_x|_g \varphi$$

  for the approximating sequence $(\boldsymbol{u}^k) \subset W^{1,\infty}(I,\mathcal{N})$ converging to $\boldsymbol{u}$ weakly in $BV(I,\mathcal{N})$

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Thank you for your attention!